MATHEMATICS

ON THE COVERING HOMOTOPY THEOREM

BY

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1. *Introduction.* The purpose of this paper is to give a new proof of W. HUEBSCH's generalization [1] of the Hurewicz-Steenrod covering homotopy theorem ([3], pp. 50–53). In Huebsch's proof, homotopies are covered piecewise by means of a well-ordered family of real-valued functions. The proof given below involves only a countable sequence of functions, a simplification made possible by invoking stronger properties of paracompactness.

Definitions and notation on fibre spaces may be found in [1]. Definitions pertaining to paracompact spaces, as well as theorems used, may be found in [2].

We state here Huebsch's generalization of the covering homotopy theorem. Proof will be given in section 3.

Theorem: *Let* $\{X, B, p, \Omega, \{\varphi_U\}_{U\in\Omega}\}$ *and* $\{X^*, B^*, p^*, \Omega^*, \{\varphi^*{}_{U^*}\}_{U^*\in\Omega^*}\}$ *be fiberings of spaces* X *and* X^*, *respectively, where the base space* B *is paracompact. Let* f: $X \to X^*$ *be admissible,* I *the closed unit interval, and* k: $B\times I \to B^*$ *a homotopy from* B *into* B^* *such that* $k(\cdot, 0)=\bar{f}$, *the map induced by* f. *Then there exists a covering homotopy* h: $X\times I \to X^*$ *of* k, *stationary with* k *and such that* $h(\cdot, 0)=f$.

2. *Lemmas on Paracompact Spaces.* A Hausdorff space X is paracompact if and only if every open cover of X has a locally finite open refinement. It is wellknown ([2], pp. 156–160) that a regular space is paracompact if and only if every open cover has an open σ-discrete refinement.

Lemma 1: *Every open cover of a paracompact space* X *has an open* σ*-discrete refinement which is locally finite.*

Proof: Let $\{V_j\}_{j\in J}$ be an open σ-discrete refinement of the given open cover; that is, $J=\bigcup_{n=1}^{\infty} J^n$, where $J^n \cap J^m=\phi$ if $n\neq m$, and $\{V_j\}_{j\in J^n}$ is discrete for each $n=1, 2, \ldots$.

For each $j\in J$, let $n(j)$ be the unique positive integer such that $j\in J^{n(j)}$, and let $V_j^*=V_j-\bigcup_{n(j')<n(j)} V_{j'}$. Clearly $\{V_j^*\}_{j\in J}$ is a σ-discrete cover of X.

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If $j, j' \in J$ and $V_j \cap V_{j'}^* \neq \phi$, then $n(j') \leqslant n(j)$. Hence $\{V_j^*\}_{j \in J}$ is locally finite. Since X is paracompact, there is an open neighborhood U of the diagonal in $X \times X$ such that the family $\{U(V_j^*)\}_{j \in J}$ of open sets is locally finite ([2], p. 158).

Let $W_j = U(V_j^*) \cap V_j$ for $j \in J$. Then $\{W_j\}_{j \in J}$ is easily seen to be the desired refinement.

Lemma 2: *Every σ-discrete open cover $\{V_j\}_{j \in J}$ of a paracompact space X is shrinkable,* i.e., *there exists a σ-discrete open refinement $\{W_j\}_{j \in J}$ such that $W_j \subset \overline{W}_j \subset V_j$ for each $j \in J$.*

Proof: Since every locally finite open cover of X is shrinkable, this follows from the above proof of Lemma 1.

To prove the theorem, we must decompose the space $B \times I$ smoothly and in an ordered way into subsets that are mapped by the given homotopy k into slicing neighborhoods of B^*. We do this by means of an increasing sequence of functions of B into I.

Lemma 3: *Let $\{G_\alpha\}$ be an open cover of the product space $B \times I$ of a paracompact space B and the closed unit interval I. Then there exists a sequence $\{\sigma_m\}_{m=0}^{\infty}$ of continuous functions from B into I satisfying the following three conditions:*

(i) *$\sigma_0(b) = 0$ and $\sigma_m(b) \leqslant \sigma_{m+1}(b)$ for all $b \in B$ and $m = 0, 1, 2, \ldots$.*

(ii) *For each $b \in B$, there exist a neighborhood W_b of b in B and a positive integer m_b such that $\sigma_{m_b}(b') = 1$ for all $b' \in W_b$.*

(iii) *Let $B_m = \{(b, t) \in B \times I \mid \sigma_{m-1}(b) \leqslant t \leqslant \sigma_m(b)\}$, $m = 0, 1, 2, \ldots$.*

Then for each $m = 1, 2, \ldots$, $B_m = (B_m \cap B_{m-1}) \cup \bigcup_{j \in J_m} C_j$ where $\{C_j\}_{j \in J_m}$ is a discrete family of closed subsets of $B \times I$ and for each $j \in J_m$, $C_j \subset$ some G_α.

Proof: For each $b \in B$ there exist an open neighborhood U_b of b in B and real numbers $0 = t_0(b) < t_1(b) < \ldots < t_{m_b}(b) = 1$ such that

$$\overline{U}_b \times [t_{i-1}(b), t_i(b)] \subset \text{some } G_\alpha, \quad 1 \leqslant i \leqslant m_b. \tag{1}$$

Since B is paracompact, by Lemmas 1 and 2, we can find families $\{V_j\}_{j \in J}$ and $\{W_j\}_{j \in J}$ of subsets of B satisfying:

(2) For each $j \in J$, $W_j \subset \overline{W}_j \subset V_j \subset \overline{V}_j \subset$ some U_b;

(3) $\{W_j\}_{j \in J}$ and $\{V_j\}_{j \in J}$ are locally finite open covers of B;

(4) $J = \bigcup_{n=1}^{\infty} J^n$ where $J^n \cap J^m = \phi$ if $n \neq m$, and for each n, $\{V_j\}_{j \in J^n}$ and $\{W_j\}_{j \in J^n}$ are discrete families.

Then for each $j \in J$, there are real numbers $0 = t_{0j} < t_{1j} < \ldots < t_{l_j, j} = 1$ such that

(5) $\overline{V}_j \times [t_{i-1,j}, t_{ij}] \subset$ some G_α, $1 \leqslant i \leqslant l_j$.

For convenience, set $t_{ij} = 1$ for $i > l_j$.

Since B is normal, for each $n = 1, 2, \ldots$ there is a Urysohn function $\sigma^n \colon B \to I$ such that

$$\sigma^n(b) = \begin{cases} 1 \text{ if } b \in \overline{\bigcup_{j \in J^n} W_j} = \bigcup_{j \in J^n} \overline{W}_j \\ 0 \text{ if } b \in B - \bigcup_{j \in J^n} V_j. \end{cases} \tag{6}$$

That $\overline{\bigcup_{j \in J^n} W_j} = \bigcup_{j \in J^n} \overline{W}_j$ follows from (4).

For each $i = 0, 1, 2, \ldots, n = 1, 2, \ldots$ define

$$\sigma_i^n(b) = \begin{cases} \min\,(\sigma^n(b), t_{ij}) \text{ if } b \in \overline{V}_j \text{ and } j \in J^n. \\ 0 \text{ if } b \in B - \bigcup_{j \in J^n} V_j. \end{cases} \tag{7}$$

The following properties of the functions σ_i^n follow easily from their definition and from (4) and (5):

(8) $\sigma_i^n \colon B \to I$ is continuous, $i = 0, 1, \ldots, n = 1, 2, \ldots$.

(9) If $j \in J^n$ and $i \geqslant l_j$, $\sigma_i^n(b) = 1$ for all $b \in \overline{W}_j$.

(10) For $i \geqslant 1$ and $n \geqslant 1$, $\sigma_i^n(b) > \sigma_{i-1}^n(b)$ implies that $b \in V_j$ for some $j \in J^n$ and $\sigma_{i-1}^n(b) \geqslant t_{i-1,j}$.

To get the desired sequence of functions, we now take maximums along finite diagonals of the doubly infinite array $\{\sigma_i^n\}$. More precisely we define the sequence of functions $\sigma_m \colon B \to I$, $m = 0, 1, 2, \ldots$, as follows:

$$\begin{cases} \sigma_0(b) = 0 \text{ for all } b \in B. \\ \sigma_1(b) = \max\,\{\sigma_0(b), \sigma_1^1(b)\} = \sigma_1^1(b). \end{cases} \tag{11}$$

In general if

$$\sigma_m(b) = \max\,\{\sigma_{m-1}(b), \sigma_i^n(b)\} \quad \text{and} \quad m > 1,$$

then

$$\sigma_{m+1}(b) = \begin{cases} \max\,\{\sigma_m(b), \sigma_{i+1}^{n-1}(b)\} \text{ if } n \neq 1 \\ \max\,\{\sigma_m(b), \sigma_n^{i+1}(b)\} \text{ if } n = 1. \end{cases} \qquad \text{for } b \in B.$$

The functions σ_m are clearly continuous. Moreover,

(12) $\sigma_{m-1}(b) \leqslant \sigma_m(b)$ for all $b \in B$, $m = 1, 2, \ldots$.

(13) For each $j \in J$, there is an m_j such that $\sigma_{m_j}(b) = 1$ for all $b \in \overline{W}_j$. This is a consequence of (9).

Suppose now that $\sigma_m = \max\,\{\sigma_{m-1}, \sigma_i^n\}$, where $m, n, i \geqslant 1$. We then have

(14) $\sigma_{m-1}(b) \geqslant \sigma_{i-1}^n(b)$ for all $b \in B$. This follows from the order in which maximums were taken.

(15) If $\sigma_m(b) > \sigma_{m-1}(b)$, then $\sigma_i^n(b) > \sigma_{i-1}^n(b)$. Therefore by (10) $b \in V_j$ for some $j \in J^n$ and $\sigma_{m-1}(b) \geqslant \sigma_{i-1}^n(b) \geqslant t_{i-1,j}$.

For $j \in J^n$, let

$$C_j = \{(b, t) \in B \times I \mid b \in \overline{V}_j \text{ and } t_{i-1,j} \leqslant \sigma_{m-1}(b) \leqslant t \leqslant \sigma_m(b)\}.$$

Then C_j is closed in $B \times I$ and $\{C_j\}_{j \in J^n}$ is a discrete family in $B \times I$. Moreover, $\sigma_{m-1}(b) \leqslant t \leqslant \sigma_m(b)$ implies that either $(b, t) \in C_j$ for some $j \in J^n$ or $\sigma_m(b) = \sigma_{m-1}(b)$.

Thus we have constructed the functions and subsets described in the conclusion of Lemma 3.

3. Proof of the Theorem. The family $\{k^{-1}(U^*)\}_{U^* \in \Omega^*}$ is an open cover of $B \times I$ and Lemma 3. may be applied. Let $\{\sigma_m\}_{m=0}^{\infty}$ be the sequence of functions with properties (i), (ii), and (iii) with respect to this open cover.

For each $m = 1, 2, \ldots$, let $H_m = \{(x, t) \in X \times I \mid (p(x), t) \in B_m\}$.

For each $j \in J_m$ let $D_j = \{(x, t) \in X \times I \mid (p(x), t) \in C_j\}$.

We then see that, for $m \geqslant 1$,

$$H_m = (H_m \cap H_{m-1}) \cup (\bigcup_{j \in J_m} D_j),$$

where H_m is closed in $X \times I$ and $\{D_j\}_{j \in J_m}$ is a discrete family of closed subsets of $X \times I$.

Also let U_j^* be a slicing neighborhood of B^* such that $k(C_j) \subset U_j^*$, and let φ_j^* be the corresponding slicing function.

Set $H_0 = X \times \{0\}$ and define $h_0(x, 0) = f(x)$. Suppose $m \geqslant 1$ and that $h_{m-1}: H_{m-1} \to X^*$ has been defined, is continuous, covers k and is stationary with k. For $(x, t) \in H_m$ define

$$h_m(x, t) = \begin{cases} \varphi_j^*(k(p(x), t), h_{m-1}(x, \sigma_{m-1}(p(x)))) & \text{if } (x, t) \in D_j \text{ for some } j \in J_m, \\ h_{m-1}(x, t) & \text{if } (x, t) \in H_m \cap H_{m-1}. \end{cases}$$

We must show that h_m is well-defined. If $(x, t) \in D_j \cap H_m \cap H_{m-1}$, then $\sigma_{m-1}(p(x)) = \sigma_m(p(x)) = t$ and so

$$\begin{aligned} h_m(x, t) &= \varphi_j^*(k(p(x), \sigma_{m-1}(p(x))), h_{m-1}(x, \sigma_{m-1}(p(x)))) \\ &= \varphi_j^*(p^*(x^*), x^*) = x^*, \text{ where } x^* = h_{m-1}(x, \sigma_{m-1}(p(x))). \end{aligned}$$

It is now easily verified from the properties of the slicing function and the induction hypothesis that h_m is well-defined, continuous, covers k and is stationary with k on H_m, and also that h_{m-1} and h_m match on $H_{m-1} \cap H_m$.

Then for $(x, t) \in X \times I$ define $h(x, t) = h_m(x, t)$ if $(x, t) \in H_m$. h is clearly well-defined and the only property that needs verification is continuity. Given $(x, t) \in X \times I$, let $b = p(x)$. By (ii) from Lemma 3, there is a neighborhood W_b of b in B and an integer m_b such that $\sigma_{m_b}(b')=1$ for all $b' \in B$.

Therefore $p^{-1}(W_b)\times I \subset \bigcup_{m=0}^{m_b} H_m$ on which h is obviously continuous. We have shown that h is continuous on some neighborhood of each point of $X\times I$. Therefore h is continuous on $X\times I$ and the proof is complete.

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REFERENCES

1. HUEBSCH, W., On the Covering Homotopy Theorem, Annals of Mathematics, **61**, 555–563 (1955).
2. KELLEY, J. L., General Topology, New York (1955).
3. STEENROD, N., The Topology of Fibre Bundles, Princeton (1951).